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Large-scale flow and spiral core instability in Rayleigh-Bénard convection

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The spiral core instability, observed in large aspect-ratio Rayleigh-Bénard convection, is studied numerically in the framework of the Swift-Hohenberg equation coupled to a large-scale flow. It is shown that the instability leads to nontrivial core dynamics and is driven by the self-generated vorticity. Moreover, the recently reported transition from spirals to hexagons near the core is shown to occur only in the presence of a nonvariational nonlinearity, and is linked to the spiral core instability. Qualitative agreement between the simulations and the experiments is demonstrated. [S1063-651X(97)51305-5]

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One of the most intriguing and unexpected recent discoveries in natural pattern formation is the experimental observation of spatiotemporally disordered spiral and target patterns [1,2] in large aspect-ratio Rayleigh-Bénard convection (RBC) in Boussinesq fluids, in a parameter range where previously only rolls were known to be stable [3]. This regime is characterized by the spontaneous and continuous emergence and annihilation of large extended spiral and target patterns. Theoretically, these novel states were successfully reproduced both by numerical simulations of the Swift-Hohenberg (SH) model coupled to a self-consistent large-scale flow [see Eqs. (1)–(3)] [4,5], as well as by the integration of the full thermally driven Navier-Stokes equations in the Boussinesq approximation [6]. It is currently postulated that the large-scale flow is necessary for the spatiotemporal chaotic state with many spirals and targets [4–7]. In a first attempt to understand these states, Cross and Tu proposed a physical mechanism based on wave-number frustration, in which defects have an invasive nature and create spirals and targets [7]. In detailed experiments by Assenheimer and Steinberg [8,9], and more recently by Plapp and Bodenschatz [10], a new instability of spiral cores of single- and multi-armed spirals was observed. The striking feature of this instability is that spiral cores oscillate periodically with a frequency considerably higher than the frequency of the overall spiral rotation [10]. For yet higher supercriticality, a novel transition from spirals to hexagons was found in which up- and down-flow hexagons may coexist [8,9]. These hexagons often invade the background RBC pattern, originating mainly from extended pattern cores.

Here, we present numerical simulations of the spiral core dynamics performed in the framework of the SH model coupled to a self-consistent large-scale flow. We show that this simple model exhibits both the spiral core instability and the spiral-to-hexagon transition, and that both are linked via the large-scale flow. In the case of the spiral core instability, we demonstrate that the velocity field generated by the spiral tip decreases the local wave number and eventually drives the tip into the Eckhaus unstable region. Phase slips then occur, locally winding the spiral up and returning the wave number into the stable domain. This instability is found to have a well-defined threshold in Rayleigh and Prandtl number space. Furthermore, the interpretation of the numerical results is supported by the analysis of a similar but simpler problem: a single-armed spiral in an external velocity field created by a point vortex located at the spiral core, in the limit of infinite Prandtl number when the large-scale flow and the order parameter are decoupled. In addition, we observed, at higher supercriticality and only in the presence of nonvariational nonlinear terms, a transition from spirals to hexagons. Both up- and down-flow hexagons are generated simultaneously near the spiral core. In our numerics, as well as in the experiments, the core oscillations always precede the transition to the hexagonal state. Below, we also present experimental data taken from a single spiral embedded in a complex pattern texture. The resemblance is rather remarkable.

We considered the well-established model, which describes RBC in a Boussinesq fluid rather successfully [11],

$$\psi_t+(\mathbf{u}\cdot\boldsymbol{\nabla})\psi=\epsilon\psi-g\psi^3+3(1-g)(\boldsymbol{\nabla}\psi)^2\nabla^2\psi-(1+\nabla^2)^2\psi, \tag{1}$$

$$\Omega_t-\sigma(\nabla^2-c^2)\Omega=g_m\hat{\mathbf{z}}\cdot\boldsymbol{\nabla}(\nabla^2\psi)\times\boldsymbol{\nabla}\psi, \tag{2}$$

$$\Omega=\boldsymbol{\nabla}\times\mathbf{u}. \tag{3}$$

Here ψ is the order parameter, $\mathbf{u}$ the horizontal velocity field of the large-scale flow, and Ω the vertical component of the vorticity. The control parameter ϵ represents the reduced Rayleigh number, while σ characterizes the Prandtl number of the fluid. The parameter g allows us to more accurately reproduce the stability properties of convection patterns, while g_m characterizes the coupling strength between the order parameter ψ and the vorticity Ω. The phenomenological parameter c is introduced to describe the local dissipation of the vorticity (e.g., due to friction at the bottom of the convection cell) [11,12]. Thus, Eq. (1) describes the dynamics of the order parameter ψ, while Eq. (2), using the definition of

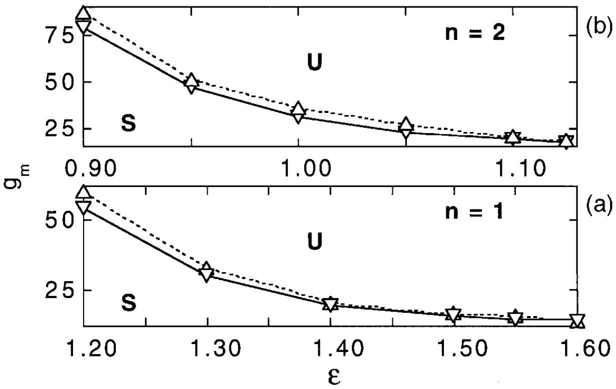

FIG. 1. Stability diagram for one- (a) and two-armed (b) spirals for $c^2=2$, $\sigma=1$, and $g=1$. As ϵ increases, the spiral core becomes unstable at the dashed line, regaining stability at the solid line as ϵ decreases.

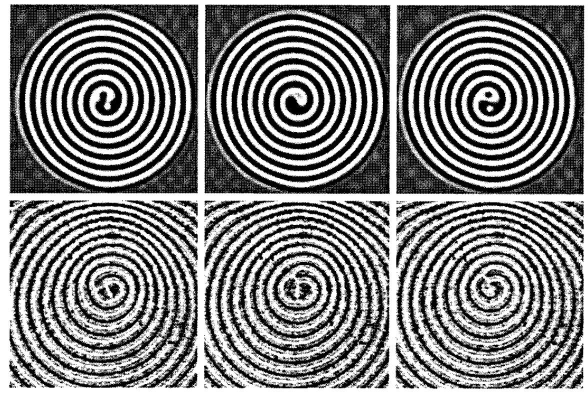

FIG. 2. Snapshots of periodic spiral core oscillations. Top row: simulations with $\epsilon=1.45$, $g=0.9$, $g_m=27$, $c^2=2$, and $\sigma=1$; time delay between frames 5 and integration domain radius $R=55$. Bottom row: experiments with $\epsilon=2.88$ and $\sigma=4.5$.

the vorticity [Eq. (3)], represents the coupling of the large-scale flow field $\mathbf{u}$ and the order parameter. For $g=1$ and $g_m=0$ Eqs. (1)–(3) reduce to the Swift-Hohenberg equation (SHE).

We solved Eqs. (1)–(3) in a domain of 256×256 mesh points using a pseudospectral method based on the fast Fourier transform. The physical domain size was typically restricted to 150×150. Circular boundary conditions were enforced by ramping ϵ towards negative values at distances $r>R_{max}=55$. The computations were performed on a parallel Cray J932 supercomputer, and verified on a 512×512 grid.

We started the simulations from initial conditions of the form $\psi=\cos(qr+n\theta)$, where r and θ are the polar coordinates, q is the wave number, and $n=\pm1,\pm2,\ldots$ is the topological charge of the spiral ($|n|$ is the number of spiral arms, while the sign corresponds to the chirality). These initial conditions relax in about 10 to 20 horizontal diffusion times to spirals. For sufficiently small values of the parameters ϵ and g_m (see Fig. 1), the spirals maintain a stable rigid rotation with an angular velocity depending both on g_m and ϵ, as well as on the topological charge n [13]. Typical spatial distributions of the order parameter and the vorticity field for one-armed spirals are shown, e.g., in Ref. [5]. The spiral tip generates a highly localized vorticity peak at the core [5]. For n-armed spirals, n identical vortices are created at the core. Here, the spiral cores are stable and only experience a slow off-center drift if the aspect ratio is not sufficiently large [12].

For ϵ above some threshold, depending on g_m (see Fig. 1), we observed a novel spiral core instability. In contrast to the off-center drift, it persists even in an infinite system, since the unstable mode is localized near the core. The main feature of the instability is that the spiral core oscillates in the reference frame of the rotating spiral. The critical values of the control parameter ϵ depend on g_m as well as on n. The bifurcation leading to core oscillations is hysteretic and its bistable region increases with g_m. Figure 1 shows the bifurcation diagram as a function of ϵ and g_m for $n=1,2$ [13]. Figure 2 presents typical numerical, as well as experimental, snapshots of the core oscillation for one-armed spirals [14]. Similar behavior occurs for two-armed spirals.

The core oscillations can be illustrated by comparing the temporal behavior of the order parameter at points near the core and at the rim. At the edge, the core oscillations are negligible so that mainly the background rotation is sensed. Figure 3 shows plots for one-armed spirals for three values of ϵ. Figure 3(a) illustrates the rigid spiral rotation, below the threshold for core oscillations. The central (solid line) and peripheral points (dashed line) oscillate with the same frequency, the unambiguous signature of the spiral's rigid rotation. Figures 3(b) and 3(c) show the spiral dynamics above criticality. Here, the core oscillates at a much higher frequency than the peripheral point. The core thus has a fast rotation in the framework of the spiral's overall rotation.

The corresponding experimental data are shown in Fig. 4. Figure 4(a) shows the rigid body rotation of a one-armed spiral below the onset of the core instability. Clearly, the oscillations near and away from the core are phase locked. Figure 4(b) presents a single burst of fast oscillations of the core above the threshold of the instability. On this short time scale the peripheral signal does not vary significantly and is not shown. Similar dynamics were reported in Refs. [8] and [10].

Our numerical simulations of the full model [Eqs. (1)–

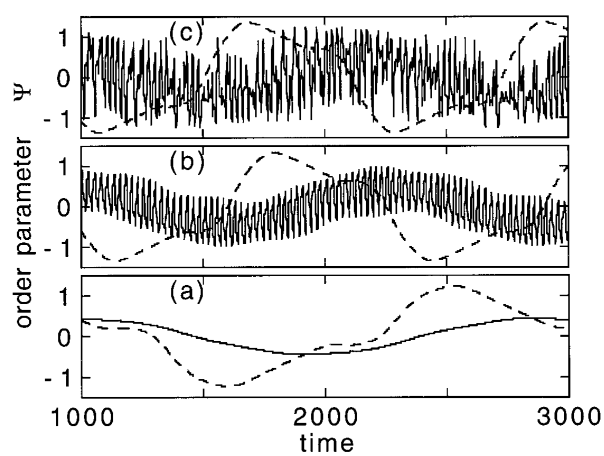

FIG. 3. Order parameter ψ at the core (solid) and periphery (dashed) of a one-armed spiral with $g=1, g_m=95, \sigma=5$, and $c^2=1$: (a) below threshold ($\epsilon=1.1$); (b) and (c) above threshold ($\epsilon=1.3$ and 1.35).

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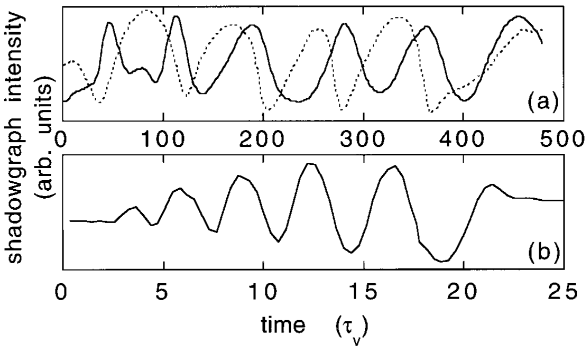

FIG. 4. Shadowgraph intensity at the core (solid) and periphery (dashed) of a one-armed spiral: (a) below threshold ($\epsilon=2.05$, $\sigma=4.5$) and (b) above threshold ($\epsilon=2.88$, $\sigma=4.5$). In (b) only the fast core signal is shown.

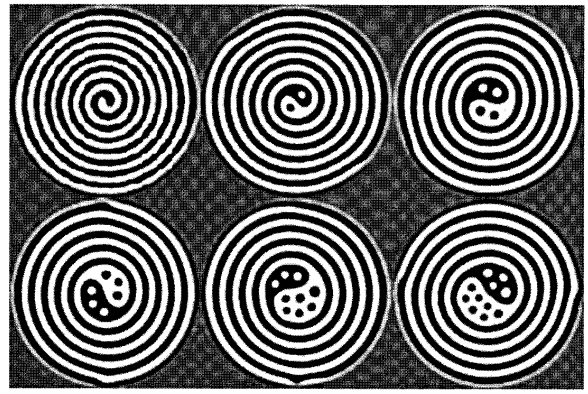

FIG. 5. Hexagon invasion from a spiral core obtained from the full model with $\epsilon=1.9$, $g=0.75$, $g_m=10$, $c^2=2$, and $\sigma=1$. Snapshots taken at $t=10$, 110, 470, 650, 340, and 2350.

(3)] suggest that the vorticity generated by the spiral tip plays a major role in the core dynamics. Therefore, the origin of the core oscillations can easily be understood—at least qualitatively—in the framework of the following simplified model. Consider a one-armed spiral solution of the form $A(r)\cos\phi$, where $\phi=qr+\theta$, in the framework of the SHE [Eq. (1)], coupled to an external velocity field generated by a fixed point vortex with circulation Γ, placed at the center of rotation. The sign of the vortex is chosen to correspond to the self-generated vortex in the full model. Using the phase approximation [i.e., neglecting the variations of the amplitude $A(r)$], we obtain (see Refs. [12] and [15]), that $\phi_t=-\Gamma/r^2\partial_\theta\phi+\cdots$. Due to the nonuniform phase rotation, the local wave number decreases linearly in time $q_t\propto-\Gamma/r^3$. In other words, the external velocity field winds the spiral up near the core. Eventually, the local wave number will be carried away from the stable band and the Eckhaus instability will be initiated. This stage can no longer be described within the phase approximation. Abrupt phase jumps by 2π (phase slips) consequently emerge and return the wave number back into the stable region. This process then recurs, leading to quasiperiodic oscillations.

We simulated the spiral dynamics using this simplified model, with a velocity profile of the form $\mathbf{u}=\Gamma r^{-1}\hat{\theta}$, and observed the above-mentioned scenario. As the magnitude of the circulation Γ increases, a bifurcation similar to the one described above occurs. At small Γ a steady rotation persists, while for $\Gamma>\Gamma_c$ the core starts to oscillate.

Recently Assenheimer and Steinberg reported a transition from the spiral and target chaotic state to a state of up- and down-flow hexagons, as the supercriticality ϵ increased [9]. These hexagons started to develop and invade the system primarily from spiral and target cores and other defects. Dewel *et al.* [16] demonstrated that in the framework of the SHE at large ϵ, the coexistence and linear stability of up- and down-flow hexagons is caused by the excitation of a *quasineutral* zero mode, which breaks the local inversion symmetry $\psi\to-\psi$. Despite their linear stability, these hexagons are nonlinearly unstable because their free energy, in the framework of the pure variational SHE ($g=1$, $g_m=0$), is higher than that of rolls. As a result, nuclei of hexagons, immersed in rolls, ultimately shrink. Because the generalized SHE [Eqs. (1)–(3)] with either $g\neq1$ and/or $g_m\neq0$ is nonvariational, a simple relative stability analysis of rolls versus hexagons becomes impossible.

Numerics performed with a nonvariational coefficient $1-g=0.25$, relatively large coupling to the vorticity $g_m=10$ (customary for this model, e.g., Ref. [12]) and supercriticality above the threshold for the spiral core instability, $\epsilon=1.9$, show that hexagons indeed invade rolls. We have observed the simultaneous nucleation of up- and down-flow hexagons at the core which subsequently spread out (see Fig. 5). Similar dynamics, obtained experimentally, is shown in Fig. 6. However, one might speculate that a local wave number change near the core, rather than the large-scale flow, causes the transition. To compare, we performed simulations for the same parameters but without a large-scale flow (i.e., $g=0.75$, $g_m=0$, and $\epsilon=1.9$). In this case spots emerge from the side wall rather than from the core region. Thus, the interaction between the large-scale flow and the spiral flow itself is required for both the spiral-to-hexagon transition as well as for the spiral core instability. Numerically as well as experimentally, it always precedes the spiral-to-hexagon transition. Based on the results of Ref. [16], we speculate that the zero mode must exist when coexisting hexagons are present. It is thus plausible to suggest that the mean flow near the core locally unwinds the wave number to zero. As such, the spiral core instability generates the zero mode of the order parameter [17], responsible for the formation of the coexisting hexagons. Although we do not fully understand

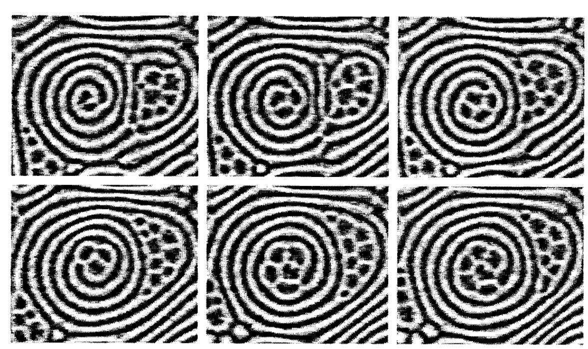

FIG. 6. Experimental hexagon nucleation at a spiral core with $\epsilon=3.19$, $\sigma=4.5$, and time delay between frames $\Delta t=3.6$, 3.6, 22.7, 18.0, and 10.7 τ_v.

the triggering mechanism, the presented observations strongly support this conjecture.

Summarizing, we studied effects of large-scale flow on the dynamics of a spiral core. Although our computations were performed in the framework of a simplified phenomenological model, two characteristic features of the dynamics were found also observed experimentally: the spiral core instability and the spiral-to-hexagon transition. The vorticity field generated at a spiral core when $g_m \neq 0$, plays a major role in the spiral oscillations. These oscillations are not observed in the variational model in which the coupling with the vorticity mode is absent (i.e., $g_m = 0$). On the other hand, very similar oscillations are observed in a model with a fixed vortex pinned at the center of rotation of the spiral. At higher supercriticality the vorticity drives the zero mode of the order parameter ψ by unwinding the spiral and prompts the hexagon formation. Then, the core oscillations may initiate the transition to the hexagonal state. Certainly, a more detailed analysis of a physically more justified (but more complicated) model, similar to that of Ref. [6], is desirable.

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[17] Large-scale flow and zero mode of the order parameter should not be equated. Physically, the latter can be related to the long-wave variation of the vertical temperature gradient from its nominal value. Note that the zero mode has different signs for up- and down-flow hexagons, the vertical vorticity has the same sign everywhere.